

# CLASSIFICATION OF EINSTEIN METRICS ON $I \times S^3$

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**ABSTRACT.** We present a complete classification of Einstein metrics on the space  $M = I \times S^3$ , where  $I$  is the interval  $(0, l)$  or  $(0, \infty)$  or their closures, and we consider separate metric functions  $f$  and  $h$  of  $t \in I$  for the base and fiber of the Hopf fibration  $S^1 \rightarrow S^3 \rightarrow S^2$ . All such metrics yielding smooth and complete manifolds are included and discussed. The results are surprisingly rich, including many well-known examples and several one-parameter families of metrics with a variety of geometries.

## 1. INTRODUCTION

Einstein manifolds in four dimensions are important in both geometry and in physics, where, as gravitational instantons, they have applications in quantum gravity (see [3] and [8]). A well-studied 4-manifold is the cylinder on the 3-sphere  $M = I \times S^3$ ,  $I$  an interval of the real line. The Hopf fibration provides a natural way of decomposing the geometry of  $S^3$  into two components, the base space  $S^2$  and fibers  $S^1$ . We define real-valued functions  $h(t)$  and  $f(t)$  such that for each  $t \in I$ , these functions determine the metric on the submerged components in  $S^3$ . We then pose and answer the question: what functions  $f$  and  $h$  yield Einstein manifolds  $(M, g)$ ?

The first example of a compact, *inhomogenous* Einstein 4-manifold (with positive scalar curvature) was given in 1978 by D. Page [8], a non-trivial fibration of  $S^2$  by  $S^2$ . This example appears in our classification in section 5.3. Soon afterwards, Bérard Bergery (see [1] and [2]) gave a generalization of Page's result to arbitrary even dimension. We concentrate on dimension four. The key to our approach is a change of independent variable that allows us to isolate  $h$  and integrate it explicitly in terms of this new variable and a small number of parameters. The bulk of the paper is an exploration of parameter space in search of solutions that yield smooth and geodesically complete Einstein manifolds. In a number of cases, we are able to integrate and obtain explicit formulas. In the non-integrable cases, we may still determine if complete non-singular manifolds are possible, allowing us to classify all possible solutions of interest.

We begin in section 2 with a discussion of the global frame and metric we use for  $M$  and a calculation of its Ricci curvature tensor in terms of the functions  $f$  and  $h$ . Restricting to Einstein metrics, we obtain ODE's that admit explicit general solutions via the change of variable mentioned above. The classification and discussion of solutions occurs in sections 3 through 6, and we conclude in section 7. Along the way we encounter a rich collection of spaces, including  $\mathbb{R}^4$ ,  $S^4$ ,

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$\mathbb{C}P^2$ ,  $TS^2$ ,  $\mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$  (Page's example), several one-parameter families of Einstein metrics, and Einstein orbifolds.

## 2. PRELIMINARIES

We begin by constructing a global frame for  $M = I \times S^3$ . Considering the quaternions  $\mathbb{H} = \mathbb{C} \times \mathbb{C}$ , a real vector space with basis  $\mathbf{1} = (1, 0)$ ,  $\mathbf{i} = (i, 0)$ ,  $\mathbf{j} = (0, 1)$ ,  $\mathbf{k} = (0, i)$ , and inner product given by  $\langle p, q \rangle = \frac{1}{2}(p^*q + q^*p)$ , where  $p^* = (\bar{a}, -b)$  for  $p = (a, b)$ . The function  $\phi : \mathbb{H} \rightarrow \mathbb{R}^4$  given by  $\phi(x^1 + x^2i, x^3 + x^4i) = (x^1, x^2, x^3, x^4)$  provides a natural smooth manifold structure on  $\mathbb{H}$ ; it is just the structure of  $\mathbb{R}^4$  obtained from  $\phi^{-1}$ . Define vector fields given at each point  $q \in \mathbb{H}$  by  $\mathcal{X}_1|_q = \mathbf{i}q$ ,  $\mathcal{X}_2|_q = \mathbf{j}q$ ,  $\mathcal{X}_3|_q = \mathbf{k}q$ , where we multiply quaternions according to  $(a, b)(c, d) = (ac - \bar{d}b, da + b\bar{c})$ , and make the canonical identification  $T_q\mathbb{H} = \mathbb{H}$ . Let  $\mathcal{S} = \{q \in \mathbb{H} \mid \langle q, q \rangle = 1\}$ , the three dimensional submanifold of unit quaternions. Since for any “imaginary”  $p \in \mathbb{H}$  (i.e.  $p^* = -p$ ),  $\langle q, pq \rangle = 0$ , the set  $\{\mathcal{X}_i|_p\}$  spans each tangent space  $T_pM$  and  $\{\mathcal{X}_i\}$  is a global frame for  $\mathcal{S}$  (see [4] p.203).

The Hopf fibration is associated with the partition of  $\mathcal{S}$  by unit-speed curves (the Hopf fibers) defined for each  $q = (a, b) \in \mathcal{S}$  by  $c_q(\tau) = (ae^{i\tau}, be^{i\tau})$ ,  $\tau \in \mathbb{R}$  (see [9] p.29). Note that  $c' = \mathbf{i}q$ , thus  $\mathcal{X}_1$  is tangent to the Hopf fiber at each point.

Now  $\phi$  restricts to a diffeomorphism  $\phi : \mathcal{S} \rightarrow S^3$  that pushes forward to a map  $\phi_* : T\mathcal{S} \rightarrow TS^3$  between tangent bundles that yields a global frame for  $S^3$ :

$$\begin{aligned} X_1 &= -x^2 \frac{\partial}{\partial x^1} + x^1 \frac{\partial}{\partial x^2} - x^4 \frac{\partial}{\partial x^3} + x^3 \frac{\partial}{\partial x^4}, \\ X_2 &= -x^3 \frac{\partial}{\partial x^1} + x^4 \frac{\partial}{\partial x^2} + x^1 \frac{\partial}{\partial x^3} - x^2 \frac{\partial}{\partial x^4}, \\ X_3 &= -x^4 \frac{\partial}{\partial x^1} - x^3 \frac{\partial}{\partial x^2} + x^2 \frac{\partial}{\partial x^3} + x^1 \frac{\partial}{\partial x^4}, \end{aligned}$$

where  $X_i = \phi_*(\mathcal{X}_i)$ . For arbitrary smooth vector fields  $V = V^i \partial/\partial x^i$ ,  $W = W^j \partial/\partial x^j$  the Lie bracket is given by  $[V, W] = VW - WV = (VW^j - WV^j) \partial/\partial x^j$ . We calculate:  $[X_1, X_2] = -2X_3$ ,  $[X_1, X_3] = 2X_2$ ,  $[X_2, X_3] = -2X_1$ . We will be interested in metrics on  $M$  of the form:

$$(1) \quad g = dt^2 + f^2(t)(\omega^1)^2 + h^2(t)[(\omega^2)^2 + (\omega^3)^2],$$

where  $\{\omega^i\}_{i=1}^3$  is the dual coframe to  $\{X_i\}$ ,  $t \in I$ , and  $f$  and  $h$  are smooth functions of  $t$ . A metric  $g$  on a smooth manifold  $M$  is an Einstein metric, and the pair  $(M, g)$  an Einstein manifold, if for all  $p \in M$  and for all  $X, Y \in T_pM$ ,

$$\text{Ric}(X, Y) = \lambda \cdot g(X, Y)$$

for some  $\lambda \in \mathbb{R}$ , the Einstein constant. The Ricci curvature tensor  $\text{Ric}$  is given by the first contraction of the Riemann curvature tensor  $R$ , or alternatively as a sum of sectional curvatures:

$$\text{Ric}(X, Y) = \sum_i g(R(e_i, X)Y, e_i),$$

where  $\{e_i\}_{i=0}^{n-1}$  is an orthonormal basis for  $T_pM$ .

We set  $e_0 = \partial/\partial t$ ,  $e_1 = f^{-1}X_1$ ,  $e_2 = h^{-1}X_2$ ,  $e_3 = h^{-1}X_3$ , and then  $\{e_i\}_{i=0}^3$  is a global, orthonormal frame for  $(M, g)$ . Note that  $[\partial/\partial t, X_i] = 0$  for  $i = 1, 2, 3$ . It

follows from this and the product rule that

$$\begin{aligned} [e_0, e_1] &= -\frac{f'}{f}e_1 & [e_0, e_2] &= -\frac{h'}{h}e_2 & [e_0, e_3] &= -\frac{h'}{h}e_3 \\ [e_2, e_3] &= -2\frac{f}{h^2}e_1 & [e_1, e_2] &= -\frac{2}{f}e_3 & [e_1, e_3] &= \frac{2}{f}e_2. \end{aligned}$$

We are now in a position to calculate the Riemann curvature tensor  $R$ , given by

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X, Y]}Z,$$

via the Koszul formula ([7] p.214):

$$\langle \nabla_X Y, Z \rangle = \frac{1}{2} \{ X \langle Y, Z \rangle + Y \langle X, Z \rangle - Z \langle X, Y \rangle - \langle Y, [X, Z] \rangle - \langle X, [Y, Z] \rangle - \langle Z, [Y, X] \rangle \},$$

where here and throughout  $\langle X, Y \rangle = g(X, Y)$  for any tangent vectors  $X$  and  $Y$ , and  $\nabla$  is the Levi-Civita connection. Working in the basis  $\{e_i\}$  the matrix  $(g_{ij})$  becomes the identity. The form of  $g$  guarantees that all the off-diagonal terms of  $(\text{Ric}_{ij})$  vanish, so for  $g$  to be an Einstein metric it is necessary and sufficient that each diagonal element of the Ricci matrix satisfy  $\text{Ric}_{ii} = \text{Ric}(e_i, e_i) = \lambda$ . Since  $X_2$  and  $X_3$  are treated identically in terms of the metric,  $\text{Ric}_{22} = \text{Ric}_{33}$ , and we have three total diagonal elements to consider.

We begin with

$$\text{Ric}_{00} = \langle R(e_0, e_0)e_0, e_0 \rangle + \langle R(e_1, e_0)e_0, e_1 \rangle + \langle R(e_2, e_0)e_0, e_2 \rangle + \langle R(e_3, e_0)e_0, e_3 \rangle.$$

Since  $R(X, X)X = 0$  for any  $X$ , the first term is zero. For the next three terms we get  $\langle R(e_1, e_0)e_0, e_1 \rangle = -f''/f$ , and  $\langle R(e_2, e_0)e_0, e_2 \rangle = \langle R(e_3, e_0)e_0, e_3 \rangle = -h''/h$ , which gives

$$\text{Ric}_{00} = -\frac{f''}{f} - 2\frac{h''}{h}.$$

Next we have

$$\text{Ric}_{11} = \langle R(e_0, e_1)e_1, e_0 \rangle + \langle R(e_2, e_1)e_1, e_2 \rangle + \langle R(e_3, e_1)e_1, e_3 \rangle.$$

We calculate  $\langle R(e_0, e_1)e_1, e_0 \rangle = -f''/f$ , and  $\langle R(e_2, e_1)e_1, e_2 \rangle = \langle R(e_3, e_1)e_1, e_3 \rangle = -(f'h'/fh) + (f^2/h^4)$ , so

$$\text{Ric}_{11} = -\frac{f''}{f} - 2\frac{f'h'}{fh} + 2\frac{f^2}{h^4}.$$

Finally,

$$\text{Ric}_{22} = \text{Ric}_{33} = \langle R(e_0, e_2)e_2, e_0 \rangle + \langle R(e_1, e_2)e_2, e_1 \rangle + \langle R(e_3, e_2)e_2, e_3 \rangle.$$

We calculate  $\langle R(e_0, e_2)e_2, e_0 \rangle = -h''/h$ ,  $\langle R(e_1, e_2)e_2, e_1 \rangle = -(f'h'/fh) + (f^2/h^4)$ , and  $\langle R(e_3, e_2)e_2, e_3 \rangle = -(h'/h)^2 - (3f^2/h^4) + (4/h^2)$ , yielding

$$\text{Ric}_{22} = \text{Ric}_{33} = -\frac{h''}{h} - \frac{f'h'}{fh} - \frac{h'^2}{h^2} + \frac{4}{h^2} - 2\frac{f^2}{h^4} = \lambda.$$

Thus the metric given by (1) is Einstein if and only if  $f$  and  $h$  satisfy the following three ordinary differential equations (see also [2] p. 274):

$$(2) \quad -\frac{f''}{f} - 2\frac{h''}{h} = \lambda,$$

$$(3) \quad -\frac{f''}{f} - 2\frac{f'h'}{fh} + 2\frac{f^2}{h^4} = \lambda,$$

$$(4) \quad -\frac{h''}{h} - \frac{f'h'}{fh} - \frac{h'^2}{h^2} + \frac{4}{h^2} - 2\frac{f^2}{h^4} = \lambda.$$

So the classification of Einstein manifolds with metrics of the type given by (1) is equivalent to a classification of solutions to (2), (3), and (4). We shall be interested in solutions of this system on a maximal interval  $I$  such that both  $f$  and  $h$  are positive. Because a singularity may occur at the endpoint(s) of  $I$ , solutions to the above system do not, in general, yield geodesically complete manifolds. However, if suitable boundary conditions are satisfied at the endpoint, the singularity can be removed by a change of coordinates.

**Smoothness and completeness.** At the endpoint  $t = 0$  we require all of  $S^3$  to collapse to a point, so  $f(0) = h(0) = 0$ . This is analogous to the origin in polar or spherical coordinates. To avoid a conical singularity we require  $f'(0) = 1$ , and  $h'(0) = 1$  or  $0$ . In cases where  $f$  and  $h$  are positive for all  $t > 0$ , there are no further restrictions on the functions. If  $f$  has positive roots we call the least root  $l$  and say the fiber collapses at  $l$ . In this case we require  $f'(l) = -1$  to have a smooth metric. Note that if  $h = 0$  at some point then we must have  $f = 0$  or we do not have a smooth 4-manifold.

A Riemannian manifold  $(M, g)$  is *geodesically complete* if every arc-length parameterized geodesic with parameter  $\tau$  is defined for all  $\tau \in \mathbb{R}$ . The only concern in our case regards the  $t$ -lines of  $M$ , which are always geodesics with arc-length parameter  $t$ .  $(M, g)$  admits a smooth completion as long as the metric functions collapse smoothly at  $t = 0$  and possibly at  $t = l$  as described above, since we can add points to  $M$  at the end points of  $I$ . In this case we'll say that  $g$  or  $M$  is smooth and complete. The geometry is incomplete if, for example,  $f \rightarrow \infty$  as  $t \rightarrow 0$ . It may happen that the above conditions are satisfied except that  $f'(0) = n$  or  $f'(0) = -f'(l) = n$  for some positive integer  $n > 1$ , in which case we can complete  $M$  to an Einstein orbifold  $I \times S^3/\mathbb{Z}_n$ , which is smooth except at the singularities at the end points, which correspond to fixed points of a  $\mathbb{Z}_n$  action. In this case we can say  $g$  or  $M$  is complete but not smooth.

**Solving the equations.** To solve the above system, we follow [2] and examine the equations (5) = (2) - (3) and (6) = (3) - (2) + 2 · (4):

$$(5) \quad \frac{h''}{h} - \frac{f'h'}{fh} + \frac{f^2}{h^4} = 0,$$

$$(6) \quad -2\frac{f'h'}{fh} - \frac{h'^2}{h^2} - \frac{f^2}{h^4} + \frac{4}{h^2} = \lambda.$$

It turns out that (5) can be integrated to yield  $f$  in terms of  $h$ :

$$(7) \quad f = \frac{|hh'|}{\sqrt{1 - ah^2}},$$

where  $a \in \mathbb{R}$  is a suitable constant. Now if  $a > 0$ , then without loss of generality we may assume  $a = 1$  by the following argument. Given a metric that is a solution

when  $a > 0$ , one can define the following rescaling

$$dt \leftrightarrow a^{-1/2} dk \quad f \leftrightarrow a^{-1/2} F \quad h \leftrightarrow a^{-1/2} H.$$

This leaves the metric unchanged in form since  $G = dk^2 + F^2(\omega^1)^2 + H^2[(\omega^2)^2 + (\omega^3)^2] = ag$ , and  $F$  and  $H$  clearly satisfy (7) since  $dh/dt = dH/dk$ . Similarly if  $a < 0$  we may assume  $a = -1$  and so our solutions break into three cases,  $a = 1, -1$ , or  $0$ .

We now introduce a new independent variable  $r$  such that

$$(8) \quad \frac{dr}{dt} = \frac{f}{h^2}.$$

Note that this implies  $dt/dr = h^2/f$  since  $r$  is evidently an increasing function on  $I$  and thus one-to-one. Equation (7) then becomes autonomous in  $h$  as

$$(9) \quad \dot{h} = \pm h \sqrt{1 - ah^2} \quad \Rightarrow \quad r = \pm \int \frac{dh}{h \sqrt{1 - ah^2}},$$

where here and throughout a dot denotes differentiation with respect to  $r$ . For the case  $a = 0$  the solution is clearly  $h = e^{\pm r}$ , and we obtain  $h = \text{sech } r$  for  $a = 1$  and  $h = \text{csch } r$  for  $a = -1$ . Note that we have suppressed the constants of integration because they may be removed by a translation of  $r$ . We now proceed to find solutions for  $f$  and discuss the resulting metrics.

### 3. SOLUTIONS WHEN $a = 0$

For this case we have  $h = \dot{h} = e^r$ . Substituting into (6) yields

$$(10) \quad f = \sqrt{(-\lambda/6)e^{4r} + e^{2r} + Ce^{-2r}},$$

where  $C \in \mathbb{R}$  is a constant of integration. The value of the Einstein constant  $\lambda$  is only geometrically significant up to sign. In fact, we may assume  $|\lambda| = 6$  or  $0$ , since we can rescale the metric as with the constant  $a$ , which is here zero. For convenience, we similarly consider cases based on the sign of  $C$  and so have 9 total subcases to consider.

3.1.  $\lambda = 0, C = 0$ . In this case  $f = h = e^r$  and

$$f' = \frac{df}{dt} = \frac{df}{dr} \frac{dr}{dt} = e^r \frac{e^r}{e^{2r}} = 1 = h',$$

so up to a translation in  $t$  the metric is  $g = dt^2 + t^2[(\omega^1)^2 + (\omega^2)^2 + (\omega^3)^2]$  and  $I = (0, \infty)$ .  $g$  is the Euclidean metric on  $\mathbb{R}^4$  in polar coordinates with  $t$  as the distance function from the origin. The singularity at  $t = 0$  can be removed by changing to rectangular coordinates.

3.2.  $\lambda = 6, C = 0$ . Let  $s = e^r$  (we use this substitution throughout), then we have  $f = \sqrt{s^2 - s^4} = s\sqrt{1 - s^2}$ ,  $\frac{dt}{ds} = s/f = (1 - s^2)^{-1/2}$ , which has solution  $s = \sin t$  and thus  $f = \sin t \cos t = (1/2) \sin 2t$ . Thus

$$(11) \quad g = dt^2 + (1/4) \sin^2 2t (\omega^1)^2 + \sin^2 t [(\omega^2)^2 + (\omega^3)^2].$$

Here  $t \in (0, \pi/2)$  and we have an explicit smooth metric on all of  $M$ , which is in fact the Fubini-Study metric on  $\mathbb{C}P^2$ . The singularity at  $t = 0$  is of the same type as in section 2.1, while at  $t = \pi/2$  each of the Hopf fibres is collapsed to a point, but this singularity can be similarly removed by a change of coordinates.

3.3.  $\lambda = -6, C = 0$ . In this similar case  $f = s\sqrt{1+s^2}$ , and  $t = \sinh^{-1} s$  so  $f = \sinh t \cosh t$ . This gives

$$(12) \quad g = dt^2 + (1/4) \sinh^2 2t (\omega^1)^2 + \sinh^2 t [(\omega^2)^2 + (\omega^3)^2],$$

and  $I = (0, \infty)$ .  $M$  is a non-compact hyperbolic dual to the preceding case  $\mathbb{CP}^2$ . The singularity is similarly removed at  $t = 0$  and  $g$  is the complex hyperbolic metric (see [6]).

3.4.  $\lambda = 0, C > 0$ . We get  $g = s^4(s^4+C)^{-1}ds^2 + (s^2+Cs^{-2})(\omega^1)^2 + s^2[(\omega^2)^2 + (\omega^3)^2]$  and  $t = \int_0^s du/\sqrt{1+Cu^{-4}}$  with  $t \in (0, \infty)$ . Note that  $f \rightarrow \infty$  as  $s \rightarrow 0$ , and since  $t \rightarrow 0$  as  $s \rightarrow 0$ , the fiber does not collapse at  $t = 0$  and the metric is not complete.

3.5.  $\lambda = 0, C < 0$ . In this case we have the same metric as above except with negative  $C$ . To check the behavior at the endpoints, first let  $D = (-C)^{1/4}$ , then  $dt/ds = s^2(s^4 - D^4)^{-1/2}$ , and

$$t(s) = \int_D^s \frac{u^2 du}{\sqrt{u^4 - D^4}} \Rightarrow f'|_{t=0} = \frac{df}{ds} / \frac{dt}{ds} \Big|_{s=D},$$

where  $t \in (0, \infty)$  and  $s \in (D, \infty)$ . We have  $f' = \frac{df}{ds} / \frac{dt}{ds} = 1 + (D/s)^4$ , so  $f'(t = 0) = 2$ ,  $h'(t = 0) = 0$  and  $h(t = 0) = D$ . After removing the singularity at  $t = 0$ , we obtain a Ricci-flat Einstein metric on the tangent bundle of the 2-sphere  $TS^2$  known as the Eguchi-Hanson metric (see [3]). The limit metric as  $D \rightarrow 0$  is the Euclidean metric on  $\mathbb{R}^4$ . Also note that for different  $D > 0$ , the metrics are homothetic.

3.6.  $\lambda = -6, C > 0$ . Here  $g = s^4(s^4 + s^6 + C)^{-1}ds^2 + (s^2 + s^4 + Cs^{-2})(\omega^1)^2 + s^2[(\omega^2)^2 + (\omega^3)^2]$ , and  $t \in (0, \infty)$ , but here  $f \rightarrow \infty$  as  $t \rightarrow 0$ , so  $g$  is not complete.

3.7.  $\lambda = -6, C < 0$ . Let  $D = -C$ , then  $g = s^4(s^4 + s^6 - D)^{-1}ds^2 + (s^2 + s^4 - Ds^{-2})(\omega^1)^2 + s^2[(\omega^2)^2 + (\omega^3)^2]$ . It is clear that  $dt/ds = s^2(s^4 + s^6 - D)^{-1/2}$  has a single positive root, call it  $z$ , so  $I = (0, \infty)$ . Here  $f' = \frac{df}{ds} / \frac{dt}{ds} = 1 + 2s^2 + Ds^{-4}$ , and for a smooth metric we require (i.)  $1 + 2z^2 + Dz^{-4} = n$ , for an integer  $n \geq 0$ . Since we are looking for complete metrics, we may assume that  $f$  vanishes at the root, so (ii.)  $z^2 + z^4 - Dz^{-2} = 0$ . Taking (i.) +  $z^{-2}$ (ii.) we obtain  $3z^2 = n - 2$ . In terms of  $z$ ,  $D = z^4 + z^6$ , and since  $h$  behaves properly as  $t \rightarrow 0$ , we have a complete Einstein metric when  $D = (1/3^2)(n - 2)^2 + (1/3^3)(n - 2)^3$  over a family of manifolds  $I \times S^3/\mathbb{Z}_n$  for integers  $n \geq 3$ .

3.8.  $\lambda = 6, C > 0$ . We have  $g = s^4(s^4 - s^6 + C)^{-1}ds^2 + (s^2 - s^4 + Cs^{-2})(\omega^1)^2 + s^2[(\omega^2)^2 + (\omega^3)^2]$  and  $I = (0, \infty)$ , but here as in cases 2.4 and 2.6,  $f \rightarrow \infty$  as  $t \rightarrow 0$ , so  $g$  cannot be complete.

3.9.  $\lambda = 6, C < 0$ . Here again let  $D = -C$ , and we have the same metric as above with  $t(s) = \int_{z_1}^s u^2 du / \sqrt{u^4 - u^6 - D}$ , and  $I = (0, l = t(z_2))$ ,  $s \in (z_1, z_2)$  and  $z_1$  and  $z_2$  are the two positive roots of  $f$ . Clearly they are both roots of  $s^4 - s^6 - D$  and hence  $D = z_1^4 - z_1^6 = z_2^4 - z_2^6$ . For a smooth metric we require  $f'$  to be an integer at the end points, and it follows that  $2 - 4z_1^2 + 2Dz_1^{-4} = n = -(2 - 4z_2^2 + 2Dz_2^{-4})$ . Substituting for  $D$  we obtain  $z_1^2 = (4 - n)/6$  and  $z_2^2 = (4 + n)/6$ , so  $n \leq 3$ . Substituting back into the equation for  $D$  we obtain  $6(4 - n)^2 - (4 - n)^3 = 6(4 + n)^2 - (4 + n)^3$ , which is not solved for  $n = 1, 2$  or  $3$ , and hence there are no complete Einstein metrics in this case.

4. SMOOTHNESS AND COMPLETENESS CRITERIA WHEN  $a \neq 0$ 

We consider the criteria for  $g$  to be complete when  $a = \pm 1$ . From (9) we know  $h = \text{sech } r$  for  $a = 1$  and  $h = \text{csch } r$  for  $a = -1$ . In both cases, substituting into (6) gives us the following metric

$$(13) \quad g = \frac{192s^4}{(s^2 + a)^2 G(s^2)} ds^2 + \frac{G(s^2)}{12s^2(s^2 + a)^2} (\omega^1)^2 + \frac{4s^2}{(s^2 + a)^2} [(\omega^2)^2 + (\omega^3)^2],$$

where  $s = e^r$  and

$$(14) \quad G(x) = (24 + 3C - 8a\lambda)(x^4 + 2ax^3) + 48x^2 + (24 - 3C - 8a\lambda)(2ax + 1).$$

To determine whether  $M$  is smooth and complete, we must analyze the behavior of  $g$  at the endpoints of  $I$ .

**Behavior of  $g$  at the endpoints.** We observe that  $f$  and  $h$  are zero or infinite only if  $s = 0$ ,  $s = 1$ ,  $s \rightarrow \infty$ , or  $G(s^2) = 0$ . When  $a = -1$ ,  $s \rightarrow 1$  corresponds to  $t \rightarrow \infty$ . To show this when  $G(0) > 0$ , we shall show that

$$\int_{1-\varepsilon}^1 \frac{8\sqrt{3}s^2}{(s^2 + a)\sqrt{G(s^2)}} ds$$

diverges for all  $\varepsilon > 0$ . Consider the integrand, which is equal to  $dt/ds$ , as the product of  $(s-1)^{-1}$  and a function that is continuous on  $[1-\varepsilon, 1]$ . By the properties of continuous functions,  $dt/ds < K(s-1)^{-1}$  for all  $s \in [1-\varepsilon, 1)$  and some  $K > 0$ . Since the integral of  $(s-1)^{-1}$  over  $[1-\varepsilon, 1)$  diverges, we conclude that the above integral diverges. This sort of argument also shows that  $s = 0$  and  $s \rightarrow \infty$  correspond to  $t = 0$  or  $t = l$ , and that any root of  $G(s^2)$  corresponds to  $t = 0$  or  $t = l$  if and only if it has a root of multiplicity of 1.

In order for  $M$  to be complete we require  $f = 0$  at  $t = 0$  and  $t = l$ , so if  $G(0) > 0$   $g$  is not complete since  $f \rightarrow \infty$  as  $s \rightarrow 0$ . Similarly,  $g$  is not complete when  $24 - 3C - 8a\lambda > 0$  since  $f \rightarrow \infty$  as  $s \rightarrow \infty$ . Otherwise  $f(t) = 0$  and  $h(t)$  is 0 or finite at  $t = 0$  and  $t = l$ , so  $g$  is complete provided  $M$  is a smooth manifold. We have  $df/dt = dh/dt = 1$  at  $s = 0$  when  $G(0) = 0$ , so  $M$  is smooth at  $s = 0$ . Furthermore, when  $24 - 3C - 8a\lambda = 0$ ,  $M$  is smooth for  $s$  near  $\infty$  since  $df/dt$  and  $dh/dt$  tend to  $-1$  as  $s \rightarrow \infty$ . Thus we must determine if  $M$  is smooth at roots of  $G(s^2)$ .

**Smoothness at one root of  $G$ .** Suppose  $I = (0, \infty)$  and let  $z > 0$  satisfy  $G(z^2) = 0$ . Since  $dh/dt = 0$  at  $s = z$ , to show that  $M$  is smooth we must show that for some integer  $n$ ,

$$(15) \quad \frac{df}{dt} = \frac{(24 + 3C - 8a\lambda)(2z^6 + 3az^4) + 48z^2 + a(24 - 3C - 8a\lambda)}{24z^2} = n.$$

Since  $G(z^2) = 0$ , whenever  $z \neq 1$  we have

$$(16) \quad C = \frac{(24 - 8a\lambda)(z^8 + 2az^6 + 2az^2 + 1) + 48z^4}{-3(z^8 + 2az^6 - 2az^2 - 1)}.$$

By substituting for  $C$  in (15) and solving for  $\lambda$ , we obtain

$$(17) \quad \lambda = \frac{(2 + n)z^4 + 4az^2 + (2 - n)}{2z^2}.$$

Thus for any given choice of  $z$  and  $n$ , there exist values for  $C$  and  $\lambda$  such that the manifold is smooth at  $s = z$ .

**Smoothness at two roots of  $G$ .** Suppose  $I = (0, l)$  and let  $z_1$  and  $z_2$  be roots of  $G(s^2)$ . We would like to show that  $df/dt = n$  at  $s = z_1$  and  $df/dt = -n$  at  $s = z_2$  for some positive integer  $n$ . Thus we shall consider the following system of equations, where  $i = 1, 2$ ,

$$\begin{aligned} G(z_i^2) &= 0, \\ \frac{df}{dt} &= (-1)^{i+1}n \text{ at } z_i. \end{aligned}$$

As before, we can now solve for  $C$  and  $\lambda$  in terms of  $z_1$  and  $n$ . Whenever  $z_1 \neq 1$ , we have

$$(18) \quad C = \frac{(24 - 8a\lambda)(z_1^8 + 2az_1^6 + 2az_1^2 + 1) + 48z_1^4}{-3(z_1^8 + 2az_1^6 - 2az_1^2 - 1)},$$

$$(19) \quad \lambda = \frac{(2 + n)z_1^4 + 4az_1^2 + (2 - n)}{2z_1^2}.$$

Now consider  $df/dt = -n$  at  $z_2$ . Observe that if we fix  $\lambda$  and solve for  $z_1$ , we have a quadratic equation in  $z_1^2$ . Thus the only non-negative solutions to this equation are  $z_1$  and  $\sqrt{\frac{2-n}{2+n}}z_1$ . By replacing  $n$  with  $-n$  in (19), we observe that  $z_2^{-1}$  is a solution to (19) for a fixed  $\lambda$ . If  $z_1 = z_2^{-1}$ , then by substituting in for  $z_1$  into (18), we obtain  $C = -C$ , thus  $C = 0$ . Otherwise,  $z_2 = \sqrt{\frac{2+n}{2-n}}z_1^{-1}$ , which we will assume for the remainder of this discussion.

We must choose  $z_1$  and  $n$  such that  $C$  is the same if we replace  $z_1$  with  $z_2$  in (18). Let  $C_1$  denote the value of  $C$  given by (18) and let  $C_2$  denote the value of  $C$  when  $z_1$  is replaced by  $z_2$ . We have

$$C_1 - C_2 = \frac{32n^3z_1^4}{3(2a(1 + az_1^2)^2 + n(1 - az_1^4)^2)}.$$

Thus  $C_1 = C_2$  if and only if  $n = 0$  or  $z_1 = 0$ . Since  $n = 0$  provides us with a root of multiplicity greater than one and  $z > 0$  by assumption, the manifold is smooth only when  $z_1 = 1$  or  $C = 0$ .

## 5. SOLUTIONS WHEN $a = 1$

For this case we have the following metric  $g$  on  $M$ ,

$$(20) \quad g = \frac{192s^4}{(s^2 + 1)^2 G(s^2)} ds^2 + \frac{G(s^2)}{12s^2(s^2 + 1)^2} (\omega^1)^2 + \frac{4s^2}{(s^2 + 1)^2} [(\omega^2)^2 + (\omega^3)^2],$$

where  $G(x)$  is given by (14) with  $a = 1$ . We may assume  $C \geq 0$  since if  $C < 0$ , we can make the change of variable  $u \leftrightarrow s^{-1}$  to produce a metric where  $C$  is replaced by  $-C$  and that is otherwise identical to  $g$ . We have the following three subcases, determined by the behavior of  $g$  at the endpoints of  $I$ .

5.1.  $24 + 3C - 8\lambda > 0$ . If  $24 - 3C - 8\lambda \geq 0$ ,  $G(s^2) > 0$  for all  $s > 0$ . If  $24 - 3C - 8\lambda < 0$ , by Descartes' Rule of Signs,  $G(s^2)$  has exactly one positive root,  $z$  and  $G(s^2) > 0$  on  $(z, \infty)$ . In both cases  $f \rightarrow \infty$  as  $s \rightarrow \infty$  and thus  $g$  is not complete. Observe that  $a = 1$  and  $\lambda \leq 0$  only in this case, so all complete solutions for  $a = 1$  occur when  $\lambda > 0$ .



5.2.  $24 + 3C - 8\lambda = 0$ . By the quadratic formula,  $G(s^2)$  has exactly one positive root,  $z$ , which satisfies  $z^2 = \frac{1}{8}C + \frac{1}{8}\sqrt{C^2 + 8C}$ . Thus  $G(s^2) > 0$  on  $(z, \infty)$  and  $I = (0, l)$  where  $f(0) = f(l) = 0$ , so  $M$  is complete and we proceed to check smoothness. Since  $df/dt = 1$  at  $s = 0$ , we require  $df/dt = -1$  at  $s = z$ . Solving produces exactly one solution at  $C = 0$  and  $\lambda = 3$ . In this case we have

$$\frac{dt}{ds} = \frac{2}{s^2 + 1} \Rightarrow s = \tan(t/2) \Rightarrow f = h = \sin(t).$$

Thus we get the explicit solution  $g = dt^2 + \sin^2(t)[(\omega^1)^2 + (\omega^2)^2 + (\omega^3)^2]$ , where  $t \in (0, \pi)$ .  $(M, g)$  is the unit 4-sphere.

5.3.  $24 + 3C - 8\lambda < 0$ . By Descartes' Rule of Signs,  $G(s^2)$  has at most two positive roots,  $z_1$  and  $z_2$ . To determine when  $G$  has two roots, we shall first determine the boundary of the regions in the  $C$ - $\lambda$ -plane where  $G$  has no roots and where  $G$  has two roots. This boundary occurs when  $G(x_0) = G'(x_0) = 0$  for some  $x_0 > 0$ . Solving gives us the following relationship,

$$C = \frac{8(3 - \lambda) [8 - (\lambda - 2\sqrt{\lambda^2 - 4\lambda})^3]}{24 + 3(\lambda - 2\sqrt{\lambda^2 - 4\lambda})^3}.$$

Let  $C_0$  denote this value for  $C$ . Since  $x_0 < 1$ ,  $G(x_0)$  decreases as  $C$  increases. Thus when  $C < C_0$ ,  $G(x_0) > 0$  and  $G$  has exactly two positive roots. When  $C \leq C_0$ ,  $G(x_0) \leq 0$ , so  $G(x) \leq 0$  for all  $x > 0$  since  $G$  has a maximum at  $x = x_0$ . In cases where two roots occur,  $g$  is defined for  $s \in (z_1, z_2)$ . Thus  $I = (0, l)$  and from section 4 we know the manifold is smooth only if  $z_1 = 1$  or  $C = 0$ . The case where  $z_1 = 1$  yields no smooth manifolds. However, when  $C = 0$ , we obtain a solution with  $df/dt = 1$  at  $s = z_1$  when  $z_1^{-1} = z_2 = \frac{1}{2}\sqrt{y} + \frac{1}{2}\sqrt{-y + 8/\sqrt{y}}$ , where  $y = 2(\sqrt[3]{1 + \sqrt{2}} + \sqrt[3]{1 - \sqrt{2}})$ . The root  $z_2$  was calculated using the formula for solving general cubic equations. In this case,  $\lambda$  is given by (19) with  $n = 1$ . The manifold  $(M, g)$  is a non-trivial  $S^2$  bundle over  $S^2$ , topologically  $\mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$ . This is the example originally given by D. Page [8].

## 6. SOLUTIONS WHEN $a = -1$

For this case we have

$$(21) \quad g = \frac{192s^4}{(s^2 - 1)^2 G(s^2)} ds^2 + \frac{G(s^2)}{12s^2(s^2 - 1)^2} (\omega^1)^2 + \frac{4s^2}{(s^2 - 1)^2} [(\omega^2)^2 + (\omega^3)^2],$$

and  $G(x)$  is given by (14) with  $a = -1$ . We may assume  $C \geq 0$  for the same reason as when  $a = 1$ . We have the following subcases.

**Some families of solutions when  $\lambda \leq 0$ .** We begin by considering three families of complete solutions.

6.1.  $24 + 3C + 8\lambda < 0$ . By Descartes' Rule of Signs, one readily determines that  $G$  has at most two positive roots. Since  $G(0) < 0$ ,  $G(1) > 0$ , and  $G(s^2) \rightarrow -\infty$  as  $s \rightarrow \infty$ , by the intermediate value theorem  $G(s^2)$  has exactly two positive roots,  $z_1$  and  $z_2$ , such that  $z_1 < 1 < z_2$ . Thus  $f > 0$  for  $s \in (z_1, 1) \cup (1, z_2)$ , and when one restricts  $s$  to either  $(z_1, 1)$  or  $(1, z_2)$ ,  $I = (0, \infty)$ . Thus  $g$  is complete provided it is smooth at  $t = 0$ .

6.2.  $24 + 3C + 8\lambda = 0$ . Since  $G$  is a quadratic polynomial, we readily determine that  $G(s^2)$  has exactly one positive root, namely

$$z^2 = \frac{-C + \sqrt{C^2 + 8C}}{8} < 1.$$

The metric  $g$  is defined for any  $s \in (z, 1) \cup (1, \infty)$ . When  $s$  is restricted to  $(z, 1)$ ,  $I = (0, \infty)$  and thus  $g$  is complete provided it is smooth at  $t = 0$ . When  $s$  is restricted to  $(1, \infty)$ ,  $I = (0, \infty)$  and  $g$  is complete.

6.3.  $3C > |24 + 8\lambda|$ . Since  $G(0) < 0$  and  $G(1) > 0$ ,  $G(s^2)$  has at least one root on  $(0, 1)$ . We know  $G$  is positive on  $(1, \infty)$  since  $G(1) > 0$  and  $G'(x) > 0$  on  $(0, 1)$ . Furthermore,  $G$  has no more than one root on  $(0, 1)$  because if a minimum occurs at  $x > 0$  such that  $G(x) < 0$ , then  $x > 1$  since

$$(22) \quad (24 + 3C + 8\lambda)(x - 1)(x + 1)^3 > \frac{1}{2}G'(x) - G(x) > 0.$$

Thus the metric  $g$  is defined for any  $s \in (z, 1) \cup (1, \infty)$ . When  $s$  is restricted to  $(z, 1)$ ,  $I = (0, \infty)$  and thus  $g$  is complete provided it is smooth at  $t = 0$ . When  $s$  is restricted to  $(1, \infty)$ ,  $g$  is not complete.

In the previous three sections, we concluded that  $g$  is complete provided it is smooth at  $t = 0$ . In fact, for each positive integer  $n$  we have a continuous family of manifolds as  $z$  ranges across  $(0, 1)$  and with  $C$  and  $\lambda$  given by (16) and (17). Also, given an integer  $n < 0$  we obtain other continuous families of manifolds dependent on  $z \in (1, \infty)$ . We require that  $z^2 \geq 1/3$  when  $n = 1$  and  $z^2 \leq 3$  when  $n = -1$ . When  $z^2 = 1/3$  and  $n = 1$ ,  $\lambda = 0$  and  $C = 10$ . We obtain no other smooth complete metrics for  $\lambda = 0$  and  $C > 8$ .

#### Other solutions when $\lambda \leq 0$ .

6.4.  $24 - 3C + 8\lambda = 0$ . Here  $G$  has no positive roots, so  $G(x) > 0$  on  $[0, \infty)$ . Thus  $g$  is defined for  $s \in (0, 1) \cup (1, \infty)$ . When  $s$  is restricted to  $(0, 1)$ ,  $I = (0, \infty)$  and  $g$  is complete. Thus we obtain a continuous family of smooth and complete manifolds dependent on  $\lambda$  and with  $C = (24 + 8\lambda)/3$ . Note that this family contains the manifold with  $\lambda = 0$  and  $C = 8$ .

We can explicitly write the metric for one manifold in this family in the case that  $C = 0$  and  $\lambda = -3$ . Restricting  $s$  to  $(0, 1)$ , we obtain

$$\frac{dt}{ds} = \frac{2}{s^2 - 1} \Rightarrow s = \tanh(t/2) \Rightarrow f = h = \sinh(t).$$

We get  $g = dt^2 + \sinh^2(t)[(\omega^1)^2 + (\omega^2)^2 + (\omega^3)^2]$ , where  $t \in (0, \infty)$ . We derive the same metric in the case that  $s$  is restricted to  $(1, \infty)$ .  $(M, g)$  is the complete hyperbolic 4-manifold with constant curvature  $-1$ .

When  $s$  is restricted to  $(1, \infty)$  and  $C > 0$ ,  $g$  is not complete since  $f \rightarrow \infty$  as  $s \rightarrow \infty$ .

6.5.  $24 - 3C + 8\lambda > 0$ . When  $s \neq 1$ , we have

$$G(s^2) = (24 + 3C + 8\lambda)(s^4 - s^2)^2 - 16\lambda s^4 + (24 - 3C + 8\lambda)(s^2 - 1)^2 > 0.$$

Thus the metric  $g$  is defined for any  $s \in (0, 1) \cup (1, \infty)$ . When  $s$  is restricted to either  $(0, 1)$  or  $(1, \infty)$ ,  $g$  is not complete.

#### Solutions when $\lambda > 0$ .

6.6.  $3C > 24 + 8\lambda$ . In this case,  $G$  may have either one or three positive roots depending on the values of  $C$  and  $\lambda$ . One root,  $z_1$ , is guaranteed and satisfies  $z_1 > 1$ . The other two roots,  $z_2$  and  $z_3$ , satisfy  $z_2 \leq z_3 < 1$ . In either case  $g$  is defined for  $s \in (z_1, \infty)$  and  $g$  is not complete when  $s$  is restricted to  $(z_1, \infty)$ .

To determine when  $G$  has three roots, we shall first determine the boundary of the regions in the  $C$ - $\lambda$ -plane where  $G$  has one root and where  $G$  has three roots. This boundary occurs when  $G(x_0) = G'(x_0) = 0$  for some  $x_0 > 0$ . Solving gives us the following relationship,

$$C = \frac{8(\lambda + 3) [8 + (\lambda - 2\sqrt{\lambda^2 + 4\lambda})^3]}{24 - 3(\lambda - 2\sqrt{\lambda^2 + 4\lambda})^3}.$$

Let  $C_0$  denote this value for  $C$ . For any fixed  $x < 1$ ,  $G(x)$  decreases as  $C$  increases. Thus when  $C > C_0$ ,  $G(x) < 0$  and  $G$  has only one positive root. When  $C \leq C_0$ ,  $G(x_0) > 0$ , so  $G$  has three positive roots since  $G$  is negative at  $s = 0$  and  $s = 1$ .

When three distinct roots occur,  $g$  is also defined for  $s \in (z_2, z_3)$ . Here  $I = (0, l)$  and  $(M, g)$  is not smooth.

6.7.  $24 - 3C + 8\lambda = 0$ . We have  $G(0) = 0$  and  $G(1) < 0$ . We find two positive roots,  $z_1$  and  $z_2$ , such that  $z_1 < 1 < z_2$  and

$$(23) \quad z_i = 1 + (-1)^i \sqrt{1 - \frac{8}{C}}, \text{ where } i = 1, 2.$$

Thus  $g$  is defined for  $s \in (0, z_1) \cup (z_2, \infty)$ . If we restrict  $s$  to  $(0, z_1)$ , then  $I = (0, l)$ . Since at  $s = 0$ ,  $df/dt = dh/dt = -1$ , we must find a  $z_1$  satisfying (23) and  $df/dt = \frac{1}{4}Cz^2 = n$  at  $s = z_1$ . This has no solutions. If we restrict  $s$  to  $(z_2, \infty)$ , we also find no solutions.

6.8.  $24 - 3C + 8\lambda > 0$ . Since  $G(0) > 0$ ,  $G(1) < 0$ , and  $G(s^2) \rightarrow \infty$  as  $s \rightarrow \infty$ ,  $G$  has at most two positive roots. Also,  $G''$  has positive roots only on  $(0, 1)$ , so  $G$  has exactly one root greater than 1. Furthermore,  $G$  has no more than one root on  $(0, 1)$  by the argument used in the case where  $3C > |24 + 8\lambda|$  and  $\lambda \leq 0$ . Thus  $g$  is defined for  $s \in (0, z_1) \cup (z_2, \infty)$ , where  $z_1$  and  $z_2$  are roots of  $G(s^2)$ . When  $s$  is restricted to either  $(0, z_1)$  or  $(z_2, \infty)$ ,  $g$  is not a complete metric.

## 7. CONCLUSION

We have been able to completely solve the problem of finding Einstein metrics on  $M$ , the cylinder over the 3-sphere, of the form given by (1). We have demonstrated that  $M$  supports a wide variety of Einstein metrics and provided explicit formulas wherever possible. While writing this paper we found the recent work [5] which also contains new Einstein metrics, though we are not sure if they are identical with ours. A thorough investigation of the geometry of these various solutions would be interesting and remains for future work.

## REFERENCES

- [1] Bergery, L. Bérard. "Sur de nouvelles variétés riemanniennes d'Einstein." Nancy: *Publications de l'Institut E. Cartan* no. 4, 160, 1982.
- [2] Besse, Arthur L.: *Einstein manifolds. Ergebnisse der Mathematik und ihrer Grenzgebiete, 3. Folge, Band 10.* Berlin: Springer, 1987.
- [3] Gibbons, G.W., and Hawking, S.W. "Classification of Gravitational Instanton Symmetries." *Commun. Math. Phys.* **66**, 291-310, 1979.
- [4] Lee, John M. *Introduction to Smooth Manifolds.* New York: Springer-Verlag, 2003.

- [5] Lü, H., Page, Don N., and Pope, C.N. “New inhomogenous Einstein metrics on sphere bundles over Einstein-Kähler manifolds.” *Phys. Lett. B* **593** 218-226, 2004.
- [6] Kobayashi, Shoshichi and Nomizu, Katsumi. *Foundations of Differential Geometry Vols. I & II*. Wiley-Interscience, 1996.
- [7] Kühnel, Wolfgang. *Differential Geometry: Curves - Surfaces - Manifolds*. Providence, RI: American Mathematical Society, 2002.
- [8] Page, Don N. “A Compact Rotating Gravitational Instanton.” *Phys. Lett. B* **79**, 235, 1978.
- [9] Walschap, Gerard. *Metric Structures in Differential Geometry*. New York: Springer-Verlag, 2004.

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